ITEP-TH-04/10

arXiv:1004.1222v1 [hep-th] 8 Apr 2010

On the instantons and the hypermultiplet mass of $\mathcal{N} = 2^*$ super Yang-Mills on S^4

Takuya Okuda[a] and Vasily Pestun[b1]

[a] *Perimeter Institute for Theoretical Physics
Waterloo, Ontario, N2L 2Y5, Canada*

[b] *Center for the Fundamental Laws of Nature
Jefferson Physical Laboratory, Harvard University
Cambridge, MA 02138, USA*

Abstract

We show that the physical $\mathcal{N} = 4$ super Yang-Mills theory on a four-sphere with an arbitrary gauge group receives no instanton contributions, by clarifying the relation between the hypermultiplet mass and the equivariant parameters of the mass-deformed theory preserving $\mathcal{N} = 2$ supersymmetry. The correct relation also implies that $\mathcal{N} = 4$ superconformal Yang-Mills theory with gauge group $SU(2)$ corresponds to Liouville theory on a torus with the insertion of a non-trivial operator, rather than the identity as have been claimed in the literature.

In this note we show that the physical $\mathcal{N} = 4$ super Yang-Mills[2] on S^4 with an arbitrary gauge group receives no instanton contributions. This follows from the correction we make to the relation between the mass and the equivariant parameters of the mass-deformed version of the $\mathcal{N} = 4$ theory preserving $\mathcal{N} = 2$ supersymmetry (the so-called $\mathcal{N} = 2^*$ theory). These parameters enter in the one-loop and the instanton contributions to the partition function discussed in [1]. The correct relation also leads to the correspondence of the $\mathcal{N} = 4$ $SU(2)$ theory with Liouville theory [2] on a torus that has a non-trivial operator, rather than the identity as have been claimed in the literature, inserted at a puncture.

The correction is summarized as follows. We follow the notation of [1] unless we state otherwise. Let $m = im_E, m_E \in \mathbb{R}$ be the notation used in Sections 1-4 of the paper to denote the hypermultiplet mass for the $\mathcal{N} = 2^*$ theory on S^4. The "holomorphic" instanton contribution coming from the South Pole of the four-sphere with radius r, discussed in Section 5 of [1], is given by the Nekrasov instanton partition function of the $\mathcal{N} = 2^*$ theory (the mass parameter $m_{\rm N}$ here was called m by Nekrasov in [3])

$$Z^{\mathcal{N}=2^*}_{\rm inst}(\epsilon_1, \epsilon_2, m_{\rm N}, ia)$$

[1] On leave of absence from ITEP, 117218, Moscow, Russia.

[2] By this we mean the four-dimensional gauge theory uniquely defined by the choice of a gauge group and a coupling constant, which is invariant under the maximal number (*i.e.* 32) superconformal charges.

with the identification of the equivariant parameters ϵ_1, ϵ_2 and $\epsilon_+ \equiv \epsilon_1 + \epsilon_2$

$$\boxed{\epsilon_1 = \epsilon_2 = \frac{1}{r} = \frac{\epsilon_+}{2} \quad \text{and} \quad m_{\rm N} = m + \frac{\epsilon_+}{2} = i m_E + \frac{\epsilon_+}{2}} \tag{1}$$

rather than $m_{\rm N} = m$ as was assumed in Section 5 of [1]. A similar statement holds for the "anti-holomorphic" instanton contribution coming from the North Pole.[3]

The main result of [1], adapted for the $\mathcal{N} = 2^*$ theory, is the formula for the expectation value of the Wilson loop W_R in the representation R of the gauge group G

$$Z_{S^4}^{\mathcal{N}=2^*} \langle W_R(C) \rangle_{\mathcal{N}=2^*;m} = \\ = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}} [da]\, e^{-\frac{4\pi^2 r^2}{g_{YM}^2}(a,a)} Z_{\text{1-loop}}^{\mathcal{N}=2^*}(ia;m) |Z_{\text{inst}}^{\mathcal{N}=2^*}(r^{-1}, r^{-1}, m + r^{-1}, ia)|^2 \,\mathrm{tr}_R\, e^{2\pi r i a}\,,$$

where $Z_{S^4}^{\mathcal{N}=2^*}$ is the full partition function and $\langle W_R(C) \rangle$ is the normalized expectation value of the half BPS Wilson loop along the equator C of S^4. We have also denoted by $Z_{\text{1-loop}}^{\mathcal{N}=2^*}$ the one-loop contribution to the partition function. The correction (1) implies, as we show below, that at $m = 0$ we obtain not just $Z_{\text{1-loop}}^{\mathcal{N}=2^*} = 1$ but also $Z_{\text{inst}}^{\mathcal{N}=2^*} = 1$ for any gauge group G. At this value of m the full 32 superconformal symmetries are restored, *i.e.*, the $\mathcal{N} = 2^*$ theory is promoted to $\mathcal{N} = 4$ super Yang-Mills. Hence we have

$$\boxed{Z_{\text{inst}}^{\mathcal{N}=4}(r^{-1}, r^{-1}, ia) = 1} \tag{2}$$

and the complete partition function of the $\mathcal{N} = 4$ theory as well as the Wilson loop expectation value are given simply by the action induced from the tree level, that is by the Hermitian Gaussian matrix model [5, 6]. Besides several issues concerning Liouville theory discussed later, this also resolves the discrepancy with the localization computation of [7], where it was shown that the $\mathcal{N} = 4$ theory reduces to the perturbative two-dimensional Yang-Mills, and then to the Hermitian Gaussian matrix model. The localization to the two-dimensional theory in [7] does not reproduce the four-dimensional instanton contributions summed into the Dedekind eta-function in [1] for the partition function of the physical $\mathcal{N} = 4$ Yang-Mills on S^4. After the correction (1), the two localization computations of the partition function of the physical $\mathcal{N} = 4$ on S^4 in [7] and in [8] agree in the conclusion that there are no four-dimensional instanton corrections to the partition funciton itself. The revised version of [8] contains the correct relation (1) as well as further discussion of its consequences.

First we explain why the correction (1) is needed. Let us consider the theory of [1] in the neighbourhood of the South Pole, which we locally treat as a theory on $\mathbb{R}^4$ to make the connection with Nekrasov's computation. The gauge fields are A_μ with $\mu = 1, \ldots, 4$. The two scalar fields (Φ_9, Φ_0) are grouped with the gauge fields into the $\mathcal{N} = 2$ vector multiplet, and sometimes we use the notation $\Phi_0 = i\Phi_{10} := i\Phi_0^E$. The scalar fields of the $\mathcal{N} = 2$ hypermultiplet are $(\Phi_5, \ldots, \Phi_8)$.

We represent the Lorentz group $SO(4)$ as $SU(2)_L \times SU(2)_R$. Our choice of the $\mathcal{N} = 2$ supersymmetry subalgebra of the $\mathcal{N} = 4$ theory, namely the splitting of the scalar fields into $(\Phi_5, \ldots, \Phi_8)$ and (Φ_9, Φ_{10}), breaks the $SO(6)$ R-symmetry of the $\mathcal{N} = 4$ theory down to $SO(4) \times SO(2) = SU(2)_L^R \times SU(2)_R^R \times U(1)^R$ in the notation of [1]. The factor $SU(2)_L^R \times U(1)^R$

[3]Relevance of the shift by $\epsilon_+/2$ was also noticed in [4].

is the classical R-symmetry of the $\mathcal{N}=2$ vector multiplet. The factor $SU(2)_R^R$ is the flavour symmetry of the hypermultiplet. (Turning on the mass on S^4 for the hypermultiplet kills the $U(1)^R$ symmetry and breaks the $\mathcal{N}=2$ R-symmetry group $SU(2)_L^R$ down to $SO(2)$ and the flavour symmetry group $SU(2)_R^R$ down to $U(1)_F$, so that the global symmetry group of the $\mathcal{N}=2^*$ theory on S^4 is $OSp(2|4)\times U(1)_F$.)

The bosonic fields of the theory naturally transform under the symmetry groups as

$$\overbrace{A_1,\ldots A_4}^{SU(2)_L\times SU(2)_R}\quad \overbrace{\Phi_5,\ldots,\Phi_8}^{SU(2)_L^R\times SU(2)_R^R}\quad \overbrace{\Phi_9,\Phi_{10}}^{U(1)^R}\,.$$

The sixteen-component fermionic field Ψ on the $\mathcal{N}=4$ theory in the convention of [1] is given in terms of four four-dimensional chiral spinors as

$$\Psi=\begin{pmatrix}\psi^L\\ \chi^R\\ \psi^R\\ \chi^L\end{pmatrix}.$$

Each of these spinors $\psi^L,\chi^R,\psi^R,\chi^L$ has four components. We summarize their transformation properties in the table:

ε	Ψ	$SU(2)_L$	$SU(2)_R$	$SU(2)_L^R$	$SU(2)_R^R$	$U(1)^R$
$*$	ψ^L	$1/2$	0	$1/2$	0	$+1/2$
0	χ^R	0	$1/2$	0	$1/2$	$+1/2$
$*$	ψ^R	0	$1/2$	$1/2$	0	$-1/2$
0	χ^L	$1/2$	0	0	$1/2$	$-1/2$
	$A_1\ldots A_4$	$1/2$	$1/2$	0	0	0
	$\Phi_5\ldots\Phi_8$	0	0	$1/2$	$1/2$	0
	Φ_9,Φ_{10}	0	0	0	0	$+1$
	parameters in [1]	0	ϵ_+	ϵ_+	$2m$	
	parameters in [3, 9]	ϵ_-	ϵ_+	ϵ_+	$2m_{\mathrm{N}}-\epsilon_+$	

We went ahead and presented the relation between values of the equivariant parameters in [1] and [3, 9] at the bottom of the table, which we are going to explain now.

Let the spinor $\varepsilon(x)$ be the parameter of the supersymmetry transformations (not to be confused with equivariant parameters), and let $\varepsilon(0)$ be the value of ε at the South Pole $x=0$. We restrict the $\mathcal{N}=4$ supersymmetry algebra to the $\mathcal{N}=2$ subalgebra by taking ε in the $+1$-eigenspace of the operator Γ^{5678}. Such a spinor is of the form

$$\varepsilon=\begin{pmatrix}*\\0\\ *\\0\end{pmatrix},$$

transforms in the spin-1/2 representation of $SU(2)_L^R$ and in the trivial representation of $SU(2)_R^R$. At the South Pole $\varepsilon(0)$ is of the right chirality, transforming non-trivially under

$SU(2)_R$ and the $\mathcal{N}=2$ R-symmetry $SU(2)^R_L$. Since equivariant rotations should keep the spinor $\varepsilon(0)$ invariant, the parameters of $SU(2)_R$ and $SU(2)^R_L$ must be equal, so that their action on $\varepsilon(0)$ is cancelled. These parameters are denoted as ϵ_+ in [3].

The equivariant parameters for the spatial rotation $SU(2)_L$ and the flavour rotation $SU(2)^R_R$ for the Nekrasov's $\mathcal{N}=2$ deformed theory in the Ω-background on $\mathbb{R}^4$ in [3] do not have to be related to each other. The parameter for $SU(2)_L$ in [3] is called ϵ_-, and the mass parameter for $SU(2)^R_R$ can be read off from the formula for the k-instanton contribution to the Nekrasov partition function $Z^{\mathcal{N}=2^*}_{\text{inst}} = \sum_{k\geq 0} q^k Z_k$ [3, 10]

$$
\begin{aligned}
Z_k = &\frac{1}{k!}\left(\frac{\epsilon_+(m_{\rm N}-\epsilon_1)(m_{\rm N}-\epsilon_2)}{\epsilon_1\epsilon_2(-m_{\rm N})(m_{\rm N}-\epsilon_+)}\right)^k \\
&\times \oint \prod_{I=1}^{k}\frac{d\phi_I}{2\pi i}\prod_{\alpha=1}^{N}\frac{(\phi_I - m_{\rm N}+\frac{1}{2}\epsilon_+ - a_\alpha)(\phi_I + m_{\rm N}-\frac{1}{2}\epsilon_+ - a_\alpha)}{(\phi_I+\frac{1}{2}\epsilon_+ - a_\alpha)(\phi_I - \frac{1}{2}\epsilon_+ - a_\alpha)} \\
&\times \prod_{I<J}\frac{\phi_{IJ}^2[\phi_{IJ}^2-\epsilon_+^2][\phi_{IJ}^2-(m_{\rm N}-\epsilon_1)^2][\phi_{IJ}^2-(m_{\rm N}-\epsilon_2)^2]}{[\phi_{IJ}^2-\epsilon_1^2][\phi_{IJ}^2-\epsilon_2^2][\phi_{IJ}^2-m_{\rm N}^2][\phi_{IJ}^2-(m_{\rm N}-\epsilon_+)^2]}
\end{aligned}
\tag{3}
$$

This can be understood as theorem (3.7) of [11] applied to the ADHM construction for the mass deformed $\mathcal{N}=4$ theory [12]. If the $\mathcal{N}=4$ theory were realized on $D3$-branes aligned in the real 1234-directions, $k\times k$ matrices B_1, B_2 would describe the complex coordinates of $D(-1)$ instantons in the 1234-directions, and $k\times k$ matrices B_3, B_4 their complex coordinates in the hypermultiplet scalar directions 5678. See, for example, [13].

According to the theorem, the four factors in the denominator given in the last line of (3) correspond respectively to the ADHM data B_1, B_2, B_3 and B_4. In (3.13) of [3] it can be explicitly seen that B_1 is acted on by ϵ_1 and B_2 is acted on by ϵ_2. These equivariant parameters correspond to the first two factors in the denominator. The parameters (ϵ_L, ϵ_R) for $SU(2)_L\times SU(2)_R$, and the ones (ϵ_1,ϵ_2) for $U(1)\times U(1)$ acting canonically on the $\mathbb{R}^4$ represented as $\mathbb{R}^2\oplus\mathbb{R}^2$, are related as

$$
\epsilon_R = \epsilon_1+\epsilon_2 = \epsilon_+,\ \epsilon_L = \epsilon_1 - \epsilon_2 = \epsilon_-. \tag{4}
$$

Similarly, if we denote the parameters which act on the matrices (B_3, B_4) by (ϵ_3,ϵ_4), and the parameters of $SU(2)^R_L\times SU(2)^R_R$ by $(\epsilon^R_L,\epsilon^R_R)$, they are related as

$$
\epsilon^R_L = -\epsilon_4-\epsilon_3,\ \epsilon^R_R = \epsilon_4-\epsilon_3. \tag{5}
$$

We can conclude from the last two factors in the denominator that $(\epsilon_3,\epsilon_4) = (-m_{\rm N}, m_{\rm N}-\epsilon_+)$, and therefore that

$$
\epsilon^R_L = \epsilon_+,\ \epsilon^R_R = 2m_{\rm N}-\epsilon_+. \tag{6}
$$

See also (2.5) of [13] and (3.6) of [14].

As a check we note that the above relations among the equivariant parameters are consistent with the numerator in the last line of (3) which according to the same theorem are associated with the ADHM equations. The ADHM equations for the hypermultiplet near the South Pole in [1] transform as the χ^L components, *i.e.*, the hypermultiplet equations are acted on by the group $SU(2)_L\times SU(2)^R_R$ with parameters $(\epsilon_L,\epsilon^R_R)$. The eigenvalues for the equivariant group action on the hypermultiplet equations are then $(\pm\frac{1}{2}(\epsilon_L+\epsilon^R_R), \pm\frac{1}{2}(\epsilon_L-\epsilon^R_R))$,

which in Nekrasov's notation evaluate to $(\pm(m_{\rm N}-\epsilon_2), \pm(-m_{\rm N}+\epsilon_1))$ and agree with the last two factors in the numerator.

In [1] the relation $M_{ij}M^{ij} = 4m^2$ between the generators M_{ij} of $SU(2)^R_R$ and the hypermultiplet mass m was used when deriving the one-loop contribution $Z^{\mathcal{N}=2^*}_{\text{1-loop}}$. This relation implies that the relevant equivariant parameters are given by $(\pm\frac{1}{2}(\epsilon_L+\epsilon^R_R), \pm\frac{1}{2}(\epsilon_L-\epsilon^R_R)) = (\pm m, \mp m)$. We conclude that the equvariant parameter ϵ^R_R for the $SU(2)^R_R$ flavour symmetry group of the hypermultiplet, which gives mass to the hypermultiplet, is

$$\begin{aligned} \epsilon^R_R &= 2m \quad \text{in [1]}, \\ \text{and} \quad \epsilon^R_R &= 2m_{\rm N} - \epsilon_+ \quad \text{in [3]}. \end{aligned} \tag{7}$$

This proves the identification (1). We remark that in the literature on Nekrasov's partition function and topological strings, the limit $\epsilon_1 = -\epsilon_2$ is often assumed, *i.e.*, $\epsilon_+ = 0$. In this special case the hypermultiplet mass, if taken by definition as the parameter for the $SU(2)$ flavour group, is unshifted in Nekrasov's notation, *i.e.*, m and $m_{\rm N}$ in (7) are equal. In the framework of [1], however, the equivariant parameters satisfy $\epsilon_1 = \epsilon_2$, hence m and $m_{\rm N}$ defined in (7) are distinct. In order to avoid confusion, one needs to be clear about what is meant by the hypermultiplet mass for the $\mathcal{N} = 2^*$ theory in the Ω-background. The equations (5.14)-(5.19) in the first version of [1] need to be corrected as $m \to m+\epsilon_+/2$. It is natural to regard m as the physical mass of the hypermultiplet in the $\mathcal{N} = 2^*$ theory, since the $\mathcal{N} = 4$ superconformal symmetry is recovered at $m = 0$.

Next we show that the $\mathcal{N} = 4$ theory with any gauge group receives no instanton contributions, *i.e.*, that $Z_k = 0$ for $k \geq 1$. Our strategy is to exhibit, for generic ϵ_1 and ϵ_2, the supersymmetries that get restored when $m_{\rm N} = \epsilon_1$ or ϵ_2 and lead to (goldstino) fermionic zero-modes in a background with an anti-self-dual gauge field. Since the gauge theory localizes to configurations with such gauge fields, Z_k should vanish as $m_{\rm N} \to \epsilon_1$ or ϵ_2. Then the instanton contributions in the $\mathcal{N} = 2^*$ theory on S^4 disappear in the $\mathcal{N} = 4$ limit $m \to 0$.

To analyze the symmetries let us consider the 5-dimensional picture from which the 4-dimensional gauge theory arises via dimensional reduction. The partition function of the 5-dimensional theory is given by

$$\text{Tr}\left[(-1)^F e^{-\beta H} g\right]$$

with

$$g = \exp\left[-\beta\left(\epsilon_- J^3_L + \epsilon_+ J^3_R + \epsilon_+ J^{R3}_L + (2m_{\rm N} - \epsilon_+) J^{R3}_R + a\right)\right],$$

using the equivariant parameters determined in (4) and (6). We have denoted by J^i_L, J^i_R, J^{Ri}_L and J^{Ri}_R $(i = 1,2,3)$ the generators of the groups $SU(2)_L$, $SU(2)_R$, $SU(2)^R_L$ and $SU(2)^R_R$, respectively. The spacetime is $S^1 \times \mathbb{R}^4$ where the circle has circumference β, and we identify fields up to symmetry transformations when going around the S^1. The definition of the Ω-background involves Lorentz (J^3_L, J^3_R), $\mathcal{N} = 2$ R-symmetry (J^{R3}_L) and global gauge transformations (a) [3]. For $\mathcal{N} = 2^*$ we also perform a flavor symmetry (J^{R3}_R) transformation for the hypermultiplet. Among the generators of the 4-dimensional $\mathcal{N} = 4$ Poincaré superalgebra, only those which commute with g remain symmetries in the Ω-background.

Let us switch to the 4-dimensional theory by taking the $\beta \to 0$ limit. It is again useful to view the bosonic symmetries $SU(2)_L \times SU(2)_R \times SU(4)^R$ (Lorentz $\times$ R-symmetry) of $\mathcal{N} = 4$ as a subgroup of $Spin(10)$ and focus on its Cartan $U(1)^5$, where each $U(1)$ rotates one factor of $\mathbb{R}^2$ in $\mathbb{R}^{10} = (\mathbb{R}^2)^5$. The supercharges Q^A_α and $\overline{Q}_{A\dot\alpha}$ $(A = 1,\ldots,4)$ of $\mathcal{N} = 4$

superalgebra form a 10-dimensional chiral spinor, so there are an even number of $U(1)$'s for which the eigenvalues $(s_1, ..., s_5)$ of the generators are $+1/2$. The left-handed supercharges Q^A_α then have an even number of $+1/2$ eigenvalues for the first two (Lorentz) $U(1)$'s and an even number of $+1/2$ eigenvalues for the last three (R-symmetry) $U(1)$'s. On the other hand the right-handed supercharges $\overline{Q}_{A\dot{\alpha}}$ have an odd number of $+1/2$ eigenvalues for the first two as well as for the last three $U(1)$'s.

Under the combined transformation g, a supercharge changes by the phase proportional to

$$\begin{aligned} & \epsilon_1 s_1 + \epsilon_2 s_2 + \epsilon_3 s_3 + \epsilon_4 s_4 \\ = \; & \epsilon_1 s_1 + \epsilon_2 s_2 - m_{\rm N} s_3 + (m_{\rm N} - \epsilon_+) s_4, \end{aligned}$$

where we have used (4)-(6). First notice that the supercharges with $s_1 = s_2 = s_3 = s_4$ are invariant. These are the left-handed supercharges preserved by the omega-background. Because the supersymmetry transformations generated by them involve only the self-dual part of the gauge flux, they are preserved by anti-self-dual instantons and do not lead to fermionic zero-modes. Thus in a generic omega-background such instantons can contribute.

On the other hand the supercharges with $s_1 = -s_2$ and $s_3 = -s_4$ are right-handed, and are restored as $m_{\rm N}$ tends to ϵ_1 $(s_1 = s_3)$ or ϵ_2 $(s_2 = s_3)$. They are then broken by anti-self-dual instantons and produce fermionic zero-modes, causing the path-integral to vanish for $k \geq 1$.

For gauge group $G = U(N)$ a more concrete way to see the vanishing of $Z_{k\geq 1}$ at $m_{\rm N} = \epsilon_1$ or ϵ_2 is to take its representation, (3.26) of [13], in terms of the N Young tableaux $Y_1, \ldots, Y_N$ labeling the poles in (3) as well as the fixed points of the equivariant action on the instanton moduli space. The formula involves the horizontal and vertical distances of the box $(i,j) \in Y_\alpha$ to the right and the bottom edges of Y_α, respectively. The fact that any non-trivial tableau necessarily contains a box with vanishing horizontal and vertical distances leads to $Z_k = 0$ for $k \geq 1$. We also checked explicitly that for $G = SO(N)$, the $k = 1$ contributions computed from the contour integrals [10] vanish as $m_{\rm N} \to \epsilon_1$ or ϵ_2. We conclude therefore that for any gauge group instanton contributions vanish when $m_{\rm N} = \epsilon_1$ or ϵ_2:

$$Z_{k\geq 1}|_{m_{\rm N}=\epsilon_1 \text{ or } \epsilon_2} = 0.$$

Consequently $\mathcal{N} = 4$ super Yang-Mills on S^4 receives no instanton contributions.

Let us now discuss what the identification (1) implies for the correspondence of $\mathcal{N} = 2^*$ $SU(2)$ Yang-Mills with Liouville theory on a torus [2]. We follow the convention of [2] and set the radius r of S^4 to one unless we note otherwise. We also define $Q = b + 1/b$.

Since the basic correspondence is motivated by the relation between the Liouville conformal block and the Nekrasov partition function, the mass parameter $m_{\rm AGT}$ of the $\mathcal{N} = 2^*$ theory, identified with the Liouville momentum and called m in [2], is equal to the mass $m_{\rm N}$ used by Nekrasov in [3]. Thus the relation

$$\boxed{m_{\rm AGT} = m + \frac{Q}{2}} \tag{8}$$

holds between $m_{\rm AGT}$ and the mass m used in [1] when $b = 1$, $Q = 2$.

As we showed above for gauge group $U(2)$ the Nekrasov partition function becomes equal to 1 when $m_{\rm N} = \epsilon_1$ or ϵ_2:

$$Z_{\rm inst}^{\mathcal{N}=2^*}(\epsilon_1, \epsilon_2, \epsilon_i, ia) = 1 \text{ for } i = 1 \text{ or } 2\,.$$

The AGT relation (3.15) of [2] then implies that the conformal block for $m_{\rm AGT} = b$ or $1/b$, for which $\Delta_{m_{\rm AGT}} \equiv m_{\rm AGT}(Q - m_{\rm AGT}) = 1$, is given by ${\mathcal{F}_\alpha}^b(q) = {\mathcal{F}_\alpha}^{1/b}(q) = 1/\prod_{i=1}^\infty (1 - q^i)$. This can be independently checked using (A.6) of [15].[4]

The one-loop contribution of the $\mathcal{N} = 2^*$ theory in [1] manifestly cancels at $m = 0$. To see this cancellation from the Liouville point of view, let us recall that the Liouville one-point function on the torus is given by

$$\langle V_{m_{\rm AGT}} \rangle_q = \int \frac{d\alpha}{2\pi} C(\alpha^*, m_{\rm AGT}, \alpha) |q^{\Delta_\alpha} {\mathcal{F}_\alpha}^{m_{\rm AGT}}(q)|^2\,.$$

By using (A.14) and various other formulas in [2], the DOZZ three-point function becomes

$$\begin{aligned} C(\alpha^*, m_{\rm AGT}, \alpha) &\propto a^2 \left| \frac{\exp\left[\gamma_{b,1/b}(2a + m_{\rm AGT} - Q)\right] \exp\left[\gamma_{b,1/b}(2a - m_{\rm AGT})\right]}{\exp\left[\gamma_{b,1/b}(2a - 1/b)\right] \exp\left[\gamma_{b,1/b}(2a - b)\right]} \right|^2 \\ &= a^2 \left| z_{\rm vector}^{\text{1-loop}}(a) z_{\rm adjoint}^{\text{1-loop}}(a, m_{\rm AGT}) \right|^2 , \end{aligned} \tag{9}$$

where we have dropped the factors independent of $\alpha = Q/2 + a$, and noted that a can be replaced by its complex conjugate $a^* = -a$ when we take an absolute value. From the first line, we see that the one-loop contributions cancel out and the 3-point function reduces to a^2 when $m_{\rm AGT} = b$ or $1/b$. Specializing to the case $b = 1$ again, we see that we need $m_{\rm AGT} = 1$ in order for the one-loop factors to drop out within the integral, so that the Hermitian Gaussian matrix model of [1, 5, 6] is recovered.

In classical Liouville theory as well as in quantum Teichmüller theory, the Liouville vertex operator $V_{m_{\rm AGT}}$ with $m_{\rm AGT} \in Q/2 + i\mathbb{R}$ creates a boundary whose geodesic length with respect to the constant curvature metric is proportional to the imaginary part [16, 17]. The $\mathcal{N} = 4$ limit $m_{\rm AGT} = 1$ with $b = 1$ then corresponds to a boundary of zero length, *i.e.*, a puncture with deficit angle 2π.

We note that the value $m_{\rm AGT} = 0$ is also special in several ways. The properly normalized Liouville correlator is a modular form of weights $(\Delta_{m_{\rm AGT}}, \Delta_{m_{\rm AGT}})$ [18], thus the gauge theory partition function is S-duality invariant at $m_{\rm AGT} = 0$. Also at this value there is no operator insertion and the one-point correlator reduces to the torus partition function, as well as the theory descends from M5-branes on a torus without defect operators. We emphasize, however, that it is only at $m = m_{\rm AGT} - 1 = 0$ that the theory restores the full $\mathcal{N} = 4$ superconformal symmetry.

Finally let us consider the 't Hooft loop T_j that is dual to the Wilson loop W_j in the spin-j representation [19, 20]. When $m_{\rm AGT} = 1$, the action of T_j on the holomorphic conformal block is given, up to a phase, by

$$a q^{-a^2} {\mathcal{F}_\alpha}^1 \to \sum_{-j \le s \le j} (a - s) q^{-(a-s)^2} {\mathcal{F}_{\alpha - s}}^1\,,$$

[4] We thank Y. Nakayama for pointing this out.

where the sum is over $s = -j, -j+1, \ldots, j$. Since ${\mathcal{F}_\alpha}^1$ does not depend on a, the prescription of [19, 20] gives the normalized expectation value of the 't Hooft loop as

$$\langle T_j \rangle = \frac{\int da\, \left(aq^{-a^2}\right)^* \sum_s (a-s)\, q^{-(a-s)^2}}{\int da\, (aq^{-a^2})^*\, aq^{-a^2}} = \sum_s e^{\frac{s^2 g^2 |\tau|^2}{4}} \left(1 + \frac{s^2 g^2 |\tau|^2}{2}\right). \tag{10}$$

Let us compare it with the expectation value of the Wilson loop

$$\langle W_j \rangle = \frac{\int da\, \left(aq^{-a^2}\right)^* \sum_s e^{4\pi i s a} aq^{-a^2}}{\int da\, (aq^{-a^2})^*\, aq^{-a^2}} = \sum_s e^{\frac{s^2 g^2}{4}} \left(1 + \frac{s^2 g^2}{2}\right).$$

Clearly $\langle T_j \rangle$ and $\langle W_j \rangle$ are exchanged under the S-duality transformation $\tau \to -1/\tau$. The final expression of (10) agrees at weak coupling with the semi-classical result of [21], including the bubbling contributions $|s| < j$. The same exact expression for $j = 1/2$ was obtained by localizing the $\mathcal{N} = 4$ theory in the 't Hooft loop background to instantons in the two-dimensional Yang-Mills theory in [7, 22]. It would also be interesting to see if the integral representation in the middle of (10), which admits a wave function interpretation, arises when the original localization technique of [1] is extended to the 't Hooft loop.

We thank Nadav Drukker, Jaume Gomis, Yu Nakayama, Jörg Teschner and in particular Yuji Tachikawa for useful discussion and correspondence. Research at the Perimeter Institute is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. The research of V.P. has been partially supported by a Junior Fellowship from the Harvard Society of Fellows, and grants NSh-3035.2008.2 and RFBR 07-02-00645.

References

[1] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," 0712.2824v1.

[2] L. F. Alday, D. Gaiotto, and Y. Tachikawa, "Liouville Correlation Functions from Four-dimensional Gauge Theories," 0906.3219.

[3] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** (2004) 831–864, hep-th/0206161.

[4] S. Shadchin, "Cubic curves from instanton counting," *JHEP* **03** (2006) 046, hep-th/0511132.

[5] J. K. Erickson, G. W. Semenoff, and K. Zarembo, "Wilson loops in N = 4 supersymmetric Yang-Mills theory," *Nucl. Phys.* **B582** (2000) 155–175, hep-th/0003055.

[6] N. Drukker and D. J. Gross, "An exact prediction of N = 4 SUSYM theory for string theory," *J. Math. Phys.* **42** (2001) 2896–2914, hep-th/0010274.

[7] V. Pestun, "Localization of the four-dimensional N=4 SYM to a two- sphere and 1/8 BPS Wilson loops," 0906.0638.

[8] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," 0712.2824.

[9] N. Nekrasov and A. Okounkov, "Seiberg-Witten theory and random partitions," hep-th/0306238.

[10] S. Shadchin, "On certain aspects of string theory / gauge theory correspondence," hep-th/0502180.

[11] G. W. Moore, N. Nekrasov, and S. Shatashvili, "Integrating over Higgs branches," *Commun. Math. Phys.* **209** (2000) 97–121, hep-th/9712241.

[12] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis, and S. Vandoren, "Multi-instanton calculus and the AdS/CFT correspondence in N = 4 superconformal field theory," *Nucl. Phys.* **B552** (1999) 88–168, hep-th/9901128.

[13] U. Bruzzo, F. Fucito, J. F. Morales, and A. Tanzini, "Multi-instanton calculus and equivariant cohomology," *JHEP* **05** (2003) 054, hep-th/0211108.

[14] G. W. Moore, N. Nekrasov, and S. Shatashvili, "D-particle bound states and generalized instantons," *Commun. Math. Phys.* **209** (2000) 77–95, hep-th/9803265.

[15] R. Poghossian, "Recursion relations in CFT and N=2 SYM theory," *JHEP* **12** (2009) 038, 0909.3412.

[16] N. Seiberg, "Notes on quantum Liouville theory and quantum gravity," *Prog. Theor. Phys. Suppl.* **102** (1990) 319–349.

[17] J. Teschner, "On the relation between quantum Liouville theory and the quantized Teichmueller spaces," *Int. J. Mod. Phys.* **A19S2** (2004) 459–477, hep-th/0303149.

[18] L. Hadasz, Z. Jaskolski, and P. Suchanek, "Modular bootstrap in Liouville field theory," 0911.4296.

[19] L. F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa, and H. Verlinde, "Loop and surface operators in N=2 gauge theory and Liouville modular geometry," 0909.0945.

[20] N. Drukker, J. Gomis, T. Okuda, and J. Teschner, "Gauge Theory Loop Operators and Liouville Theory," 0909.1105.

[21] J. Gomis, T. Okuda, and D. Trancanelli, "Quantum 't Hooft operators and S-duality in N=4 super Yang-Mills," 0904.4486.

[22] S. Giombi and V. Pestun, "The 1/2 BPS 't Hooft loops in N=4 SYM as instantons in 2d Yang-Mills," 0909.4272.